

COMPLETE COTORSION PAIRS IN EXACT CATEGORIES

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ABSTRACT. We discuss a generalized version of Quillen’s small object argument in arbitrary categories. We use it to give a criterion for the construction of complete cotorsion pairs in arbitrary exact categories, which is a generalization of the recent result due to Saorín and Šťovíček. This criterion also allows us to recover Gillespie’s recent work on the relative derived categories of Grothendieck categories.

1. INTRODUCTION

Ever since L. Salce introduced the notion of a cotorsion pair in the late 1970’s [21], the significance of complete cotorsion pairs (see Definition 3.1 for precise description) has been widely understood in approximation theory of modules [10], especially in the proof of Flat Cover Conjecture [1]. The study of complete cotorsion pairs in exact categories starts from the discovery of a wonderful relationship between them and the Quillen model structures on exact categories by M. Hovey [14], see also [2, 7].

One fundamental result on complete cotorsion pairs of modules is due to Eklof and Trlifaj: any cotorsion pair cogenerated by a set of modules is complete [6], which is a generalization of a corresponding result of Göbel and Shelah on abelian groups in [9]. In [14], M. Hovey extended this result to Grothendieck categories, and in [22], Saorín and Šťovíček generalized it further to exact categories with “nice” properties (see Definition 2.6 of [22]).

Both Hovey’s proof and that of Saorín and Šťovíček’s rely on Quillen’s small object argument, which is the main tool for constructing factorizations of morphisms in model categories ([11, 12, 19]). In its original form, the small object argument is only applicable to categories with small colimits. In order to use it in their setting, Saorín and Šťovíček proposed a generalized version of this argument. However, in

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practice, one preliminary condition of their version of the small object argument is that the transfinite compositions of inflations in their exact categories exist and are still inflations, and there are interesting exact categories which do not satisfy this condition even those categories are cocomplete. For example, given a Grothendieck category \mathcal{G} with a fixed generator G , there is a relative exact category \mathcal{G}_G with exact structure given by G -exact sequences, i.e. the short exact sequences in \mathcal{G} which remain exact after applying $\text{Hom}_{\mathcal{G}}(G, -)$. In this exact category \mathcal{G}_G , the transfinite compositions of inflations are not necessarily inflations if G is not finitely presentable. This motivates us to give a more generalized small object argument in Theorem 2.7 with least hypothetical conditions.

Applying this generalized small object argument, we give a method in Theorem 3.6 for constructing a complete cotorsion pair from a set of inflations in an arbitrary exact category, which is a generalization of the corresponding results of [22] and [14]. For the sake of studying model structures on exact categories, we give a handy version of Theorem 3.6 in weakly idempotent complete exact categories (see the paragraph preceding the statement of the theorem) in Theorem 3.7.

We apply Theorem 3.7 to Gillespie's construction of the G -derived categories of Grothendieck categories. In Corollary 4.2, we show that Gillespie's main result in Section 3 of [8] can be obtained directly from Theorem 3.7, which is another motivation of this paper.

The contents of the paper are as follows: In Section 2, we recall the necessary notions for stating Quillen's small object argument and prove our generalized small object argument in Theorem 2.7. Section 3 is devoted to the proofs of Theorem 3.6 and Theorem 3.7. As an application, we illustrate our results in Section 4 by recovering Gillespie's work on the relative derived category of a Grothendieck category in [8].

Throughout this paper, all colimits in concern are small colimits.

2. QUILLEN'S SMALL OBJECT ARGUMENT

In this section we recall some necessary notions from [12, Chapter 2] and [11, Chapter 10] firstly. Then we give Quillen's small object argument in a general form. For the definitions and basic properties of ordinals and cardinals, we refer the reader to [5, Chapters 4 and 5].

2.1. Relative I -cell complexes. Let \mathcal{C} be a category. Given a commutative diagram in \mathcal{C} of the following form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & \nearrow h & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

a *lift* or *lifting* in the diagram is a morphism $h : B \rightarrow X$ such that $hi = f$ and $ph = g$. A morphism $i : A \rightarrow B$ is said to have the *left lifting property* (LLP) with respect to another morphism $p : X \rightarrow Y$ and p is said to have the *right lifting property* (RLP) with respect to i if a lift exists in any diagram of the above form.

Definition 2.1. ([12, Definition 2.1.7]) Let I be a set of morphisms in a category \mathcal{C} .

(1) A morphism is *I -injective* if it has the RLP with respect to every morphism in I . The class of I -injective morphisms is denoted $I\text{-inj}$.

(2) A morphism is an *I -cofibration* if it has the LLP with respect to every I -injective morphism. The class of I -cofibrations is denoted $I\text{-cof}$.

An ordinal λ is often viewed as a category where there is a unique morphism from α to β if and only if $\alpha \leq \beta$ in λ .

Suppose \mathcal{C} is a category and λ is an ordinal. A functor $X : \lambda \rightarrow \mathcal{C}$ (i.e., a diagram

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_\alpha \rightarrow \cdots \quad (\alpha < \lambda)$$

in \mathcal{C}) is called a λ -sequence if for every limit ordinal $\gamma < \lambda$ the colimit $\text{colim}_{\alpha < \gamma} X_\alpha$ exists and the induced morphism $\text{colim}_{\alpha < \gamma} X_\alpha \rightarrow X_\gamma$ is an isomorphism.

If a colimit of a λ -sequence X exists, the morphism $X_0 \rightarrow \text{colim}_{\alpha < \lambda} X_\alpha$ is called the *transfinite composition* of X .

If \mathcal{D} is a class of morphisms in a category \mathcal{C} and λ is an ordinal, a λ -sequence of morphisms in \mathcal{D} is a λ -sequence $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_\alpha \rightarrow \cdots$ ($\alpha < \lambda$) in \mathcal{C} such that each morphism $X_\alpha \rightarrow X_{\alpha+1}$ is in \mathcal{D} for $\alpha + 1 < \lambda$. Then a *transfinite composition of morphisms in \mathcal{D}* is the transfinite composition of a λ -sequence $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_\alpha \rightarrow \cdots$ ($\alpha < \lambda$) of morphisms in \mathcal{D} . We say that a morphism $f : X \rightarrow Y$ is a *push out of a morphism in \mathcal{D}* if there exists a push out

$$\begin{array}{ccc} A & \longrightarrow & X \\ d \downarrow & & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

in \mathcal{C} with $d \in \mathcal{D}$.

Definition 2.2. ([12, Definition 2.1.9]) Let I be a set of morphisms in a category \mathcal{C} . Assume that the transfinite compositions of pushouts of morphisms in I exist. A *relative I -cell complex* is a transfinite composition of pushouts of morphisms in I . That is, if $f : A \rightarrow B$ is a relative I -cell complex, then there is an ordinal λ and a λ -sequence $X : \lambda \rightarrow \mathcal{C}$ such that f is the transfinite composition of X and such that, for each α such that $\alpha + 1 < \lambda$, there is a pushout as follows,

$$\begin{array}{ccc} C_\alpha & \longrightarrow & X_\alpha \\ g_\alpha \downarrow & & \downarrow \\ D_\alpha & \longrightarrow & X_{\alpha+1} \end{array}$$

such that $g_\alpha \in I$.

The collection of relative I -cell complexes is denoted $I\text{-cell}$. Note that $I\text{-cell}$ contains all isomorphisms. If \mathcal{C} has an initial object 0 , an object $A \in \mathcal{C}$ is an *I -cell complex* if the morphism $0 \rightarrow A \in I\text{-cell}$. The collection of I -cell complexes is denoted $\text{Cell}(I)$.

Lemma 2.3. *Let \mathcal{C} be an arbitrary category and I be a set of morphisms in \mathcal{C} . If the transfinite compositions of pushouts of morphisms in I exist, then $I\text{-cell} \subseteq I\text{-cof}$.*

Proof. Assume that we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ f \downarrow & & \downarrow j \\ D & \xrightarrow{h} & B \end{array}$$

such that $j \in I\text{-inj}$ and f is a relative I -cell complex.

Let $f : C \rightarrow D$ be the transfinite composition of the λ -sequence $X : \lambda \rightarrow \mathcal{C}$:

$$C = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \cdots \rightarrow X_\alpha \xrightarrow{f_\alpha} X_{\alpha+1} \rightarrow \cdots \quad (\alpha + 1 < \lambda)$$

with the f_α being a pushout of a morphism in I . Let $\tau_\alpha : X_\alpha \rightarrow D = \text{colim}_{\alpha < \lambda} X_\alpha$ be the colimit morphism for all $\alpha < \lambda$. We will construct the morphism $u_\alpha : X_\alpha \rightarrow A$ by transfinite induction such that $ju_\alpha = h\tau_\alpha$ and $u_{\alpha+1}f_\alpha = u_\alpha$. Let $u_0 = g$. Assume that $X_0 \rightarrow X_1$ is the pushout of $i : E \rightarrow F$ in I :

$$\begin{array}{ccccc}
E & \longrightarrow & X_0 & \xrightarrow{g} & A \\
\downarrow i & & \downarrow f_0 & \nearrow \tau & \downarrow j \\
F & \xrightarrow{\quad \psi \quad} & X_1 & \xrightarrow{h\tau_1} & B.
\end{array}$$

Since $j \in I\text{-inj}$, there is a lifting $v : F \rightarrow A$ which induces a morphism $u_1 : X_1 \rightarrow A$ such that $ju_1 = h\tau_1$ and $u_1f_0 = g$ by the universal property of pushouts. Assume that we have defined $u_\alpha : X_\alpha \rightarrow A$ for all $\alpha < \beta$. If β is a limit ordinal, let $u_\beta : X_\beta = \text{colim}_{\alpha < \beta} X_\alpha \rightarrow A$ be the induced morphism by u_α for $\alpha < \beta$, then $ju_\beta = h\tau_\beta$. If β has a predecessor α , then replace $f_0 : X_0 \rightarrow X_1$ by f_α in the case of $\alpha = 0$, we can construct a morphism $u_{\alpha+1} : X_{\alpha+1} \rightarrow A$ satisfying $u_{\alpha+1}f_\alpha = u_\alpha$ and $ju_{\alpha+1} = h\tau_{\alpha+1}$ which completes our transfinite induction. Therefore, let $u : D = \text{colim}_{\alpha < \lambda} X_\alpha \rightarrow A$ be the induced morphism by the u_α , then $ju = h$ and $ugh = g$ by the universal property of colimits. \square

Lemma 2.4. *Let \mathcal{C} be a category and I a set of morphisms in \mathcal{C} . If the transfinite compositions of pushouts of morphisms in I exist. Then the transfinite compositions of I -cell exist and are still in I -cell.*

Proof. Let λ be an ordinal and $X : \lambda \rightarrow \mathcal{C}$ be a λ -sequence

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_\alpha \rightarrow \cdots \quad (\alpha < \lambda)$$

such that each morphism $X_\alpha \rightarrow X_{\alpha+1}$ for $\alpha + 1 < \lambda$ is the transfinite composition of the γ_α -sequence

$$X_\alpha = W_0^\alpha \rightarrow W_1^\alpha \rightarrow W_2^\alpha \rightarrow \cdots \rightarrow W_\beta^\alpha \rightarrow \cdots \quad (\beta < \gamma_\alpha)$$

of pushouts of morphisms in I . By *interpolating* (see Definition 10.2.11 of [11]) these sequences for all $\alpha < \lambda$ into the λ -sequence X , we get a μ -sequence $Y : \mu \rightarrow \mathcal{C}$ of pushouts of morphisms in I ; see Propositions 10.2.8 and 10.2.13 of [11], or the proof of Lemma 2.1.12 of [12]. By assumption, the transfinite composition of the μ -sequence Y exists, that is, $\text{colim}_{\gamma < \mu} Y_\gamma$ exists. By the construction of Y , we have $\text{colim}_{\alpha < \lambda} X_\alpha = \text{colim}_{\gamma < \mu} Y_\gamma$ and the transfinite composition $X_0 \rightarrow \text{colim}_{\alpha < \lambda} X_\alpha$ is the transfinite composition $Y_0 \rightarrow \text{colim}_{\gamma < \mu} Y_\gamma$ which is a relative I -cell complex. \square

2.2. The small object argument. Recall that, the *cofinality* of a limit ordinal λ , denoted by $\text{cf}(\lambda)$, is the smallest cardinal κ such that there exists a subset T of λ with $|T| = \kappa$ and $\sup(T) = \lambda$.

Let \mathcal{C} be a category. Let κ be a cardinal and \mathcal{D} a class of morphisms in \mathcal{C} . An object A of \mathcal{C} is said to be κ -small relative to \mathcal{D} if for every ordinal λ with $\text{cf}(\lambda) > \kappa$

and every λ -sequence $X : \lambda \rightarrow \mathcal{C}$ of morphisms in \mathcal{D} , the natural morphism

$$\operatorname{colim}_{\alpha < \lambda} \operatorname{Hom}_{\mathcal{C}}(A, X_\alpha) \rightarrow \operatorname{Hom}_{\mathcal{C}}(A, \operatorname{colim}_{\alpha < \lambda} X_\alpha)$$

is an isomorphism. An object A in \mathcal{C} is called *small relative to \mathcal{D}* if it is κ -small relative to \mathcal{D} for some cardinal κ .

For simplicity, we introduce the following notion. We remind the reader that our notion has the different meaning with the corresponding notions in [11] and [4].

Definition 2.5. Let \mathcal{C} be a category. A set I of morphisms in \mathcal{C} is said to *admit the small object argument* if the following conditions hold:

- (i) The transfinite compositions of pushouts of morphisms in I exist.
- (ii) The coproducts of morphisms in I exist.
- (iii) The domains of the morphisms of I are small relative to I -cell.

Lemma 2.6. *Let I be a set of morphisms of a category \mathcal{C} .*

(a) *If the transfinite compositions of pushouts of morphisms in I exist, then any pushouts of morphisms in I -cell exist and are in I -cell.*

(b) *If I satisfies the conditions (i) and (ii) of Definition 2.5, then any pushouts of coproducts of morphisms in I exist and are in I -cell.*

Proof. (a) Assume that $f : A \rightarrow B$ is a relative I -cell complex. Then there is an ordinal λ and a λ -sequence $X : \lambda \rightarrow \mathcal{C}$:

$$A = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \rightarrow X_\beta \xrightarrow{f_\beta} X_{\beta+1} \rightarrow \cdots (\beta + 1 < \lambda)$$

such that every f_β is a pushout of a morphism in I and f is the transfinite composition of X . Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g_0 \downarrow & & \\ E_0 & & \end{array}$$

be any diagram in \mathcal{C} .

We can construct a commutative diagram by transfinite induction

$$(*) \quad \begin{array}{ccccccc} X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \longrightarrow & X_\beta & \xrightarrow{f_\beta} & \cdots \\ g_0 \downarrow & & g_1 \downarrow & & & & g_\beta \downarrow & & \\ E_0 & \xrightarrow{h_0} & E_1 & \xrightarrow{h_1} & \cdots & \longrightarrow & E_\beta & \xrightarrow{h_\beta} & \cdots \end{array}$$

such that each square diagram is a pushout. In fact, since f_0 is a pushout of a morphism in I , say $i : C \rightarrow D$, we have a pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & D \\ s \downarrow & & \downarrow t \\ X_0 & \xrightarrow{f_0} & X_1 \end{array}$$

By assumption, the pushouts of morphisms in I exist, so the pushout of i along g_0s exists:

$$\begin{array}{ccc} C & \xrightarrow{i} & D \\ g_0s \downarrow & & \downarrow t \\ E_0 & \xrightarrow{h_0} & E_1 \end{array}$$

which induces a pushout diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & X_1 \\ g_0 \downarrow & & \downarrow g_1 \\ E_0 & \xrightarrow{h_0} & E_1. \end{array}$$

Assume that we have defined E_α and $g_\alpha : X_\alpha \rightarrow E_\alpha$ for all $\alpha < \beta$. If β is a limit ordinal, define $E_\beta = \text{colim}_{\alpha < \beta} E_\alpha$ which exists by assumption, and define g_β to be the morphism induced by the g_α . If β has a predecessor α , i.e. $\beta = \alpha + 1$, define $E_\beta = E_\alpha \amalg_{X_\alpha} X_{\alpha+1}$, and g_β to be the pushout of g_α along f_α (note that the pushout of g_α along f_α exists; see the case of $\alpha = 0$). Therefore we have a λ -sequence $E : \lambda \rightarrow \mathcal{C}$:

$$E_0 \xrightarrow{h_0} E_1 \xrightarrow{h_1} \cdots E_\beta \xrightarrow{h_\beta} E_{\beta+1} \rightarrow \cdots \quad (\beta + 1 < \lambda),$$

its transfinite composition $E_0 \rightarrow \text{colim}_{\alpha < \lambda} E_\alpha$ exists by assumption and is in I -cell by construction. The commutative diagram (*) induces a desired pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B = \text{colim}_{\alpha < \lambda} X_\alpha \\ g_0 \downarrow & & \downarrow \\ E_0 & \longrightarrow & \text{colim}_{\alpha < \lambda} E_\alpha \end{array}$$

in \mathcal{C} .

(b) By condition (ii) of Definition 2.5, the coproducts of morphisms in I exists. Assume that $\coprod_{s \in S} i_s : \coprod C_s \rightarrow \coprod D_s$ is a coproduct of morphisms in I indexed by a set S . By condition (i) of Definition 2.5, the pushouts of morphisms in I exist, so the coproduct $\coprod_{s \in S} i_s$ is a transfinite composition of pushouts of the i_s by Proposition

10.2.7 of [11]. In particular, it is in I -cell, and thus its pushout exists and is still in I -cell by (a). \square

Now we can prove the following generalized Quillen's small object argument:

Theorem 2.7. (The small object argument). *Let \mathcal{C} be an arbitrary category and I be a set of morphisms in \mathcal{C} . Suppose that I admits the small object argument. Then every morphism $f : X \rightarrow Y$ in \mathcal{C} admits a factorization $f = \delta(f)\gamma(f)$, where $\gamma(f) \in I$ -cell and $\delta(f) \in I$ -inj.*

Proof. Since I admits the small object argument, we can define the relative I -cell complexes. By Lemma 2.4, the transfinite compositions of relative I -cell complexes exist and are still in I -cell. By Lemma 2.6(b), the pushouts of coproducts of morphisms in I exist and are in I -cell. Moreover, the domains of the morphisms of I are small relative to I -cell by assumption. Therefore, the proof of Theorem 2.1.14 of [12] works here. \square

Remark 2.8. (1) If the category \mathcal{C} is cocomplete, the first two conditions in Definition 2.5 automatically hold for any set I of morphisms in \mathcal{C} , and we can get Proposition 10.5.16 of [11] and Theorem 2.1.14 of [12] by Theorem 2.7.

(2) Given a set I of morphisms in \mathcal{C} , assume that there is a class $\mathcal{M} \supseteq I$ of morphisms in \mathcal{C} satisfying the following conditions:

- Arbitrary pushouts of morphisms in \mathcal{M} exist and are in \mathcal{M} .
- Arbitrary transfinite compositions of morphisms in \mathcal{M} exist and are in \mathcal{M} .
- Arbitrary coproducts of morphisms in \mathcal{M} exist and are in \mathcal{M} .

Then the first two conditions in Definition 2.5 hold, and if the domains of morphisms in I are small relative to I -cell, we get Proposition 2.1 of [22].

(3) If \mathcal{C} has arbitrary coproducts and I is a set of morphisms in \mathcal{C} which admits the small object argument. By Lemma 2.4, Lemma 2.6 and Proposition 10.2.7 of [11], it can be shown that the class I -cell satisfies the conditions in (2). In this case, our theorem is equivalent to Proposition 2.1 of [22].

(4) Our theorem shows that Quillen's small object argument only depends on the properties of the chosen set I of morphisms in \mathcal{C} .

3. COMPLETE COTORSION PAIRS IN EXACT CATEGORIES

3.1. Cotorsion pairs in exact categories. The concept of an exact category is due to D. Quillen [20], a simple axiomatic description can be found in [16, Appendix

A]. Roughly speaking an *exact category* is an additive category \mathcal{A} equipped with a class \mathcal{E} of *kernel-cokernel sequences* $A \xrightarrow{s} B \xrightarrow{t} C$ in \mathcal{A} such that s is the kernel of t and t is the cokernel of s . The class \mathcal{E} satisfies exact axioms, for details, we refer the reader to [3, Definition 2.1]. Given an exact category \mathcal{A} , we will call a kernel-cokernel sequence *short exact* if it is in \mathcal{E} . The morphism s in a short exact sequence $A \xrightarrow{s} B \xrightarrow{t} C$ is called an *inflation* and the morphism t is called a *deflation*.

Given an exact category $(\mathcal{A}, \mathcal{E})$, we can define the Yoneda Ext bifunctor $\text{Ext}_{\mathcal{A}}^1(A, B)$. It is the abelian group of equivalence classes of short exact sequences $B \rightarrow C \rightarrow A$ in \mathcal{E} ; see [17, Chapter XII.4] for details.

Definition 3.1. [7, Definition 2.1] Let \mathcal{A} be an exact category. A *cotorsion pair* in \mathcal{A} is a pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects of \mathcal{A} such that

- (i) $\mathcal{X} = {}^{\perp_1}\mathcal{Y} := \{X \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^1(X, Y) = 0, \forall Y \in \mathcal{Y}\}$.
- (ii) $\mathcal{Y} = \mathcal{X}^{\perp_1} := \{Y \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^1(X, Y) = 0, \forall X \in \mathcal{X}\}$.

The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if the following conditions are satisfied:

- (iii) $(\mathcal{X}, \mathcal{Y})$ has *enough projectives*, i.e. for each $A \in \mathcal{A}$ there exists a short exact sequence $Y \rightarrow X \rightarrow A$ such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.
- (iv) $(\mathcal{X}, \mathcal{Y})$ has *enough injectives*, i.e. for each $A \in \mathcal{A}$ there exists a short exact sequence $A \rightarrow Y \rightarrow X$ such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *cogenerated by a set* if there is a set \mathcal{S} of objects in \mathcal{A} such that $\mathcal{Y} = \mathcal{S}^{\perp_1}$.

3.2. I -cell complexes. Recall the Definition 2.9 of [22]: Given a class \mathcal{S} of objects in an exact category \mathcal{A} , an object A of \mathcal{A} is called \mathcal{S} -*filtered*¹ if the morphism $0 \rightarrow A$ is the transfinite composition of a λ -sequence $X : \lambda \rightarrow \mathcal{A}$:

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \rightarrow X_\beta \xrightarrow{i_\beta} X_{\beta+1} \rightarrow \cdots (\beta + 1 < \lambda)$$

such that each i_β is an inflation with a cokernel in \mathcal{S} . The λ -sequence X is called an \mathcal{S} -*filtration* of A , and the class of all \mathcal{S} -filtered object is denoted by $\text{Filt-}\mathcal{S}$.

The following Lemma is called Eklof's Lemma [10, Lemma 3.1.2]. Its proof given here is taken from the proof of Lemma 6.2 of [14].

Lemma 3.2. *Let \mathcal{A} be an exact category. Then ${}^{\perp_1}A$ is closed under ${}^{\perp_1}A$ -filtered objects for any object $A \in \mathcal{A}$.*

¹In [14], A is called a transfinite extension of \mathcal{S} .

Proof. Suppose that B is ${}^{\perp_1}A$ -filtered, and $X : \lambda \rightarrow \mathcal{A}$ is a ${}^{\perp_1}A$ -filtration of B :

$$0 = X_0 \xrightarrow{i_0} X_1 \rightarrow \cdots \rightarrow X_\alpha \xrightarrow{i_\alpha} X_{\alpha+1} \rightarrow \cdots \quad (\alpha + 1 < \lambda)$$

such that each i_α is an inflation and $\text{coker } i_\alpha \in {}^{\perp_1}A$ for all $\alpha + 1 < \lambda$. Denote B by $X_\lambda = \text{colim}_{\alpha < \lambda} X_\alpha$, we will show that $X_\beta \in {}^{\perp_1}A$ for all $\beta \leq \lambda$ by transfinite induction.

By hypothesis, $X_0 \in {}^{\perp_1}A$. Now assume that the assertion holds for an ordinal $\beta < \lambda$, that is, $X_\alpha \in {}^{\perp_1}A$ for all $\alpha < \beta$. If β has a predecessor α , i.e. $\beta = \alpha + 1$, then we have a short exact sequences:

$$X_\alpha \xrightarrow{i_\alpha} X_{\alpha+1} \twoheadrightarrow \text{coker } i_\alpha.$$

By assumption, both X_α and $\text{coker } i_\alpha$ are in ${}^{\perp_1}A$. So $X_{\alpha+1} \in {}^{\perp_1}A$ since ${}^{\perp_1}A$ is closed under extensions.

If β is a limit ordinal, take any element $A \xrightarrow{f} N \xrightarrow{p} X_\beta \in \text{Ext}_{\mathcal{A}}^1(X_\beta, A)$. By pulling back this conflation through the morphism $X_\alpha \rightarrow X_\beta$ for all $\alpha < \beta$, we get a compatible collection of conflations, three of which are shown in the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & N & \xrightarrow{p} & X_\beta \\ \parallel & & \uparrow & & \uparrow \\ A & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha \\ \parallel & & \downarrow j_\alpha & & \downarrow i_\alpha \\ A & \xrightarrow{f_{\alpha+1}} & N_{\alpha+1} & \xrightarrow{p_{\alpha+1}} & X_{\alpha+1}. \end{array}$$

We will construct splitting $s_\alpha : X_\alpha \rightarrow N_\alpha$ of p_α such that $j_\alpha s_\alpha = s_{\alpha+1} i_\alpha$ by transfinite induction on α . This will then show that $\text{Ext}_{\mathcal{A}}^1(X_\beta, A) = 0$, as required to complete the first transfinite induction.

Since $\text{Ext}_{\mathcal{A}}^1(X_\alpha, A) = 0$, it follows that there is a splitting $t_\alpha : X_\alpha \rightarrow N_\alpha$ of p_α . We begin our transfinite induction by taking $s_0 = t_0$. If $\gamma < \beta$ is a limit ordinal, take $s_\gamma : X_\gamma = \text{colim}_{\alpha < \gamma} X_\alpha \rightarrow N_\gamma$ to be the morphism such that the following diagram commutes:

$$\begin{array}{ccccc} X_{\alpha'} & \longrightarrow & X_\alpha & \longrightarrow & X_\gamma \\ \downarrow s_{\alpha'} & & \downarrow s_\alpha & & \downarrow s_\gamma \text{ } \exists! \\ N_{\alpha'} & \longrightarrow & N_\alpha & \xrightarrow{k_\alpha} & N_\gamma. \end{array}$$

If γ has a predecessor α , i.e. $\gamma = \alpha + 1$, then we have

$$p_{\alpha+1}(j_\alpha s_\alpha - t_{\alpha+1} i_\alpha) = 0.$$

Therefore, there is a morphism $h : X_\alpha \rightarrow A$ such that $f_{\alpha+1}h = j_\alpha s_\alpha - t_{\alpha+1}i_\alpha$. Since i_α is an inflation and $\text{Ext}_{\mathcal{A}}^1(\text{coker } i_\alpha, A) = 0$, there is a morphism $g : X_{\alpha+1} \rightarrow Y$ such that $gi_\alpha = h$. Now let $s_{\alpha+1} = t_{\alpha+1} + f_{\alpha+1}g$. Then $s_{\alpha+1}$ is a section of p_α such that $j_\alpha t_\alpha = s_{\alpha+1}i_\alpha$, as required. \square

Recall that given a class \mathcal{X} of objects in an exact category \mathcal{A} , a morphism $p_A : X \rightarrow A$ with $X \in \mathcal{X}$ is called a *right \mathcal{X} -approximation*² of the object A if the induced map $\text{Hom}_{\mathcal{A}}(X', X) \rightarrow \text{Hom}_{\mathcal{A}}(X', A)$ is surjective for all $X' \in \mathcal{X}$. The subcategory \mathcal{X} is said to be *contravariantly finite* in \mathcal{A} if each object A of \mathcal{A} has a right \mathcal{X} -approximation. Dually, one can define a *left \mathcal{X} -approximation* of an object A in \mathcal{A} and \mathcal{X} is called *covariantly finite* in \mathcal{A} if any object of \mathcal{A} has a left \mathcal{X} -approximation.

Given a set I of inflations in an exact category, we use $\text{Cok}I$ to denote the class of objects

$$\{A \in \mathcal{A} \mid A \cong \text{coker}(i), \text{ for some } i \in I\}.$$

Lemma 3.3. *Let \mathcal{A} be an exact category and I be a set of inflations in \mathcal{A} . Assume that I admits the small object argument. Then*

- (a) $\text{Cell}(I)$ is contravariantly finite in \mathcal{A} .
- (b) $\text{Cell}(I) \subseteq \text{Filt-Cok}I$.
- (c) $(\text{Cok}I)^{\perp_1} = (\text{Cell}(I))^{\perp_1} = (\text{Filt-Cok}I)^{\perp_1}$.

Proof. (a) By Theorem 2.7, the small object argument, given any object $A \in \mathcal{A}$, we can decompose the morphism $0 \rightarrow A$ as $0 \xrightarrow{f} E \xrightarrow{g} A$, where $f \in I\text{-cell}$ and $g \in I\text{-inj}$. Therefore $E \in \text{Cell}(I)$. For any morphism $h : Z \rightarrow A$ with $Z \in \text{Cell}(I)$, then $0 \rightarrow Z \in I\text{-cell}$, there exists a morphism $u : Z \rightarrow E$ such that $gu = h$ since $I\text{-cell} \subseteq I\text{-cof}$ by Lemma 2.3.

(b) Assume that $A \in \text{Cell}(I)$, then $0 \rightarrow A \in I\text{-cell}$ is a transfinite composition of pushouts $X_\alpha \xrightarrow{j_\alpha} X_{\alpha+1}$ of morphisms of I . Note that each morphism j_α is an inflation with cokernel isomorphic to an object in $\text{Cok}I$. Therefore, $A \in \text{Filt-Cok}I$.

(c) Since $I \subset I\text{-cell}$ and $I\text{-cell}$ is closed under pushouts by Lemma 2.6(a), we know that $\text{Cok}I \subseteq \text{Cell}(I)$. By (b), we have $\text{Cell}(I) \subseteq \text{Filt-Cok}I$. So we only need to show that $(\text{Cok}I)^{\perp_1} \subseteq (\text{Filt-Cok}I)^{\perp_1}$. Given any $A \in (\text{Cok}I)^{\perp_1}$, then $\text{Cok}I \subseteq {}^{\perp_1}A$. But ${}^{\perp_1}A$ is closed under ${}^{\perp_1}A$ -filtered objects by Lemma 3.2, so $\text{Filt-Cok}I \subseteq {}^{\perp_1}A$ and then $A \in (\text{Filt-Cok}I)^{\perp_1}$. \square

²In the literature this is sometimes called an \mathcal{X} -precover.

Lemma 3.4. *Let \mathcal{A} be an exact category and I be a set of morphisms in \mathcal{A} . Assume that the transfinite compositions of pushouts of morphisms in I exist. Then the coproducts of $\text{Cell}(I)$ exist and are still in $\text{Cell}(I)$.*

Proof. Let $\{A_i\}_{i \in S}$ be a class of objects in $\text{Cell}(I)$ indexed by a set S . By choosing a well ordering of the set S , we can identify S with an ordinal λ . As in the proof of Proposition 10.2.7 of [11], we can define a functor $X : \lambda + 1 \rightarrow \mathcal{A}$ such that for $\alpha + 1 \leq \lambda$ the morphism $X_\alpha \rightarrow X_{\alpha+1}$ is a split inflation in I -cell by transfinite induction on α . We begin our transfinite induction by letting $X_0 = 0$. Suppose that we have constructed X_α for $\alpha < \gamma$. If γ is a successor ordinal, i.e., there is $\alpha < \lambda$ such that $\gamma = \alpha + 1$, we define $X_\gamma = X_\alpha \coprod A_\alpha$ via the pushout diagram

$$\begin{array}{ccc} 0 & \longrightarrow & A_\alpha \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & X_{\alpha+1}. \end{array}$$

Since $0 \rightarrow A_\alpha$ is a relative I -cell complex, we know that $X_\alpha \rightarrow X_{\alpha+1}$ is also a relative I -complex since I -cell is closed under pushouts by Lemma 2.6(a). By the universal property of pushouts, the morphism $X_\alpha \rightarrow X_{\alpha+1}$ is also a split inflation. If γ is a limit ordinal, by Lemma 2.4, $\text{colim}_{\alpha < \gamma} X_\alpha$ exists, so we can define $X_\gamma = \text{colim}_{\alpha < \gamma} X_\alpha$. Therefore, by transfinite induction on $\alpha \leq \lambda$, we construct a $\lambda + 1$ -sequence $X : \lambda + 1 \rightarrow \mathcal{A}$

$$0 = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{\alpha+1} \rightarrow \cdots \quad (\alpha + 1 < \lambda + 1)$$

with the morphisms in the sequence being split inflations in I -cell. By Lemma 2.4 again, the transfinite composition $0 \rightarrow \text{colim}_{\alpha \leq \lambda} X_\alpha$ exists and belongs to I -cell. In particular, $\text{colim}_{\alpha \leq \lambda} X_\alpha \in \text{Cell}(I)$. By the construction of X , $\text{colim}_{\alpha \leq \lambda} X_\alpha = \coprod_{\alpha \leq \lambda} A_\alpha$. \square

3.3. Completeness of cotorsion pairs.

Definition 3.5. [22, Definition 2.3] Let \mathcal{A} be an exact category and let I be a set of inflations. I is said to be *homological* if for any object $A \in \mathcal{A}$, the morphism $A \rightarrow 0$ belongs to I -inj implies $A \in (\text{Cok} I)^{\perp_1}$.

Recall that in an exact category \mathcal{A} , a class \mathcal{T} of objects of \mathcal{A} is called a *class of generators* of \mathcal{A} if any object $A \in \mathcal{A}$, there is a deflation $\coprod_{s \in S} G_s \twoheadrightarrow A$ with $G_s \in \mathcal{T}$ and S a set.

The following theorem is a generalization of Theorem 2.13 of [22], here we do not need the assumption that the exact category is efficient (see Definition 2.6 of [22],

one essential ingredient of an efficient exact category is that transfinite compositions of inflations exist and are still inflations).

Theorem 3.6. *Let \mathcal{A} be an exact category. Let I be a homological set of inflations which admits the small object argument. Assume that relative I -cell complexes are inflations. Then:*

(a) *For each object $A \in \mathcal{A}$, there is a short exact sequence*

$$A \twoheadrightarrow T \rightarrow B$$

with $T \in (\text{Cok}I)^{\perp_1}$ and $B \in \text{Cell}(I)$. In particular, $(\text{Cok}I)^{\perp_1}$ is covariantly finite in \mathcal{A} .

(b) *If $\text{Cell}(I)$ is a class of generators, then $({}^{\perp_1}((\text{Cok}I)^{\perp_1}), (\text{Cok}I)^{\perp_1})$ is a complete cotorsion pair in \mathcal{A} .*

(c) *If ${}^{\perp_1}((\text{Cok}I)^{\perp_1})$ is closed under coproducts and is a class of generators of \mathcal{A} , then for each object $A \in \mathcal{A}$ there is a short exact sequence*

$$T' \twoheadrightarrow B' \rightarrow A$$

with $T' \in (\text{Cok}I)^{\perp_1}$ and $B' \in {}^{\perp_1}((\text{Cok}I)^{\perp_1})$. In particular, $({}^{\perp_1}((\text{Cok}I)^{\perp_1}), (\text{Cok}I)^{\perp_1})$ is a complete cotorsion pair in \mathcal{A} .

(d) *If $\text{Filt-Cok}I$ is closed under coproducts, extensions and is a class generators of \mathcal{A} , then ${}^{\perp_1}((\text{Cok}I)^{\perp_1})$ consists precisely of direct summands of $\text{Cok}I$ -filtered objects.*

Proof. (a) Since I admits a small object argument, we can factor $A \rightarrow 0$ as the composition $A \xrightarrow{f} T \xrightarrow{g} 0$, where f is a relative I -cell complex and g is in $I\text{-inj}$. By assumption, f is an inflation and $T \in (\text{Cok}I)^{\perp_1}$. Since $I\text{-cell}$ is closed under pushouts by Lemma 2.6(a), we know that $0 \rightarrow B = \text{coker } f$ is a relative I -complex, that is to say $B \in \text{Cell}(I)$. By Lemma 3.3(c), we know that $B \in {}^{\perp_1}((\text{Cok}I)^{\perp_1})$ which implies f is a left $(\text{Cok}I)^{\perp_1}$ -approximation of A . So $(\text{Cok}I)^{\perp_1}$ is covariantly finite in \mathcal{A} .

(b) By Lemma 3.3(c), $\text{Cell}(I) \subseteq (\text{Cok}I)^{\perp_1}$, thus by (a), we only need to show that the cotorsion pair $({}^{\perp_1}((\text{Cok}I)^{\perp_1}), (\text{Cok}I)^{\perp_1})$ has enough projectives. By Lemma 3.4, $\text{Cell}(I)$ is closed under coproducts. Thus for any object $A \in \mathcal{A}$, there exists a deflation $p : C \twoheadrightarrow A$ with $C \in \text{Cell}(I)$. Let $K = \ker p$. By (a), there is a deflation $K \twoheadrightarrow T \twoheadrightarrow B$ with $T \in (\text{Cok}I)^{\perp_1}$ and $B \in \text{Cell}(I)$. By Proposition 2.12 of [3], we have a commutative diagram of exact rows and columns:

$$A \longrightarrow B$$

$$\begin{array}{ccccc}
K & \xrightarrow{\quad} & T & \twoheadrightarrow & B \\
\downarrow & & \downarrow & & \parallel \\
C & \xrightarrow{\quad} & C' & \twoheadrightarrow & B \\
\downarrow p & & \downarrow & & \\
A & \xlongequal{\quad} & A & &
\end{array}$$

such that the upper-left square is a pushout diagram. Since $\text{Cell}(I) \subseteq {}^{\perp_1}((\text{Cok}I)^{\perp_1})$ by Lemma 3.3(c), by the short exact sequence $C \rightarrowtail C' \twoheadrightarrow B$ we know that $C' \in {}^{\perp_1}((\text{Cok}I)^{\perp_1})$. Then $T \rightarrowtail C' \twoheadrightarrow A$ is a short exact sequence satisfying $T \in (\text{Cok}I)^{\perp_1}$ and $C' \in {}^{\perp_1}((\text{Cok}I)^{\perp_1})$ which shows that the cotorsion pair $({}^{\perp_1}((\text{Cok}I)^{\perp_1}), (\text{Cok}I)^{\perp_1})$ has enough projectives.

The proofs of (c) and (d) are similarly to the proof of (b). \square

By Hovey's correspondence; see [14, Theorem 2.2] and [7, Theorem 3.3], complete cotorsion pairs in an exact category \mathcal{A} is closely related to the model structures on \mathcal{A} when \mathcal{A} is weakly idempotent complete. Recall that an exact category is called *weakly idempotent complete* (WIC, for short) if every split monomorphism (a morphism with left inverse) is an inflation. For WIC exact categories, we have the following result:

Theorem 3.7. *Let \mathcal{A} be a WIC exact category. Let I be a homological set of inflations which admits the small object argument. Assume that relative I -cell complexes are inflations and $\text{Cell}(I)$ is a class of generators of \mathcal{A} . Then:*

(a) *For each object $A \in \mathcal{A}$, there is a short exact sequence*

$$T \rightarrowtail B \twoheadrightarrow A$$

with $T \in (\text{Cok}I)^{\perp_1}$ and $B \in \text{Cell}(I)$.

(b) *The cotorsion pair $({}^{\perp_1}((\text{Cok}I)^{\perp_1}), (\text{Cok}I)^{\perp_1})$ is complete and ${}^{\perp_1}((\text{Cok}I)^{\perp_1})$ consists precisely of direct summands of objects of $\text{Cell}(I)$.*

Proof. (a) Since I admits the small object argument, for any object $A \in \mathcal{A}$, we can factor the morphism $0 \rightarrow A$ as $0 \xrightarrow{f} B \xrightarrow{g} A$, where $f \in I\text{-cell}$ and $g \in I\text{-inj}$. Therefore $B \in \text{Cell}(I)$. Since $\text{Cell}(I)$ is a class of generators and it is closed under coproducts by Lemma 3.4, there exists a deflation $p : U \twoheadrightarrow A$ with $U \in \text{Cell}(I)$, i.e. $0 \rightarrow U \in I\text{-cell}$. By Lemma 2.3, p factors through g , thus g is also a deflation; see Proposition 7.6 of [3]. Let $T = \ker g$, then $T \rightarrow 0$ is a pullback of g . Since

I -inj is closed under pullback, we know that $T \rightarrow 0 \in I$ -inj. By assumption, I is homological, so $T \in (\text{Cok} I)^{\perp_1}$.

(b) By Theorem 3.6(b), the cotorsion pair $({}^{\perp_1}((\text{Cok} I)^{\perp_1}), (\text{Cok} I)^{\perp_1})$ is complete. For $A \in {}^{\perp_1}((\text{Cok} I)^{\perp_1})$, the short exact sequence $T \rightarrow B \rightarrow A$ in (a) splits. So A is a direct summand of $B \in \text{Cell}(I)$. \square

4. THE RELATIVE DERIVED CATEGORIES OF GROTHENDIECK CATEGORIES

Throughout this section we fix a Grothendieck category \mathcal{G} with a chosen set of generators $\{G_s\}_{s \in S}$. Let $G = \bigoplus_{s \in S} G_s$. Then G itself is a generator of \mathcal{G} .

4.1. The G -exact category. A short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in \mathcal{G} is called G -exact if $\text{Hom}_{\mathcal{G}}(G, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{G}}(G, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{G}}(G, C)$ is a short exact sequence of abelian groups. The morphism f in a G -exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a G -inflation and the morphism g is called a G -deflation. Let \mathcal{E}_G be the class of all G -exact sequences. Then $(\mathcal{G}, \mathcal{E}_G)$ is an exact category; see Corollary 3.4 of [8]. We will use \mathcal{G}_G to denote this exact category. We use $G\text{-Ext}_{\mathcal{G}}^1$ to denote the Yoneda Ext on \mathcal{G}_G to distinguish it from the usual Ext-functor on \mathcal{G} .

We denote by $\text{Ch}(\mathcal{G})$ the category of all chain complexes of the form

$$X : \cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots$$

over \mathcal{G} and morphisms are chain morphisms.

Given an object $A \in \mathcal{G}$, define $S_n(A) \in \text{Ch}(\mathcal{G})$ by $S_n(A)^n = A$ and $S_n(A)^k = 0$ for $k \neq n$. Similarly, define $D_n(A)$ by

$$D_n(A)^k = \begin{cases} 0 & k \neq n, n+1, \\ A & k = n, n+1. \end{cases}$$

The differential d^n in $D_n(A)$ is the identity. Note that there is an evident degreewise-split monomorphism $S_{n+1}(A) \rightarrow D_n(A)$ for each $n \in \mathbb{Z}$.

An short exact sequence $X \rightarrow Y \rightarrow Z$ of complexes in $\text{Ch}(\mathcal{G})$ is called *degreewise G -exact* if $X^n \rightarrow Y^n \rightarrow Z^n$ is G -exact for each $n \in \mathbb{Z}$. We use $\text{Ch}(\mathcal{G})_G$ to denote the category of chain complexes over \mathcal{G} with the degreewise G -exact structure. We use $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1$ to denote the Yoneda Ext on $\text{Ch}(\mathcal{G})_G$ to distinguish it from the usual Ext-functor on $\text{Ch}(\mathcal{G})$.

Remark 4.1. In general, the transfinite compositions of G -inflations are not necessarily G -inflations if the generator G is not finitely presented, i.e. $\text{Hom}_{\mathcal{G}}(G, -)$ does not necessarily preserve colimits.

4.2. The G -derived category. By Construction 1.5 of [18], see also [15, Sections 11-12], the derived category $\mathcal{D}(G)$ of the exact category \mathcal{G}_G is defined as the Verdier quotient

$$\mathcal{D}(\text{Ch}(G)) = \mathcal{K}(\mathcal{G}_G) / \mathcal{K}_{ac}(\mathcal{G}_G),$$

where $\mathcal{K}(\mathcal{G}_G)$ is the homotopy category of chain complexes over \mathcal{G}_G and $\mathcal{K}_{ac}(\mathcal{G}_G)$ is the full subcategory of $\mathcal{K}(\mathcal{G}_G)$ consisting of G -acyclic complexes: a complex X in $\text{Ch}(\mathcal{G})_G$ is G -acyclic if its differentials each factors as $X^n \xrightarrow{p_n} Z^n(X) \xrightarrow{i_n} X^{n+1}$ where p_n is a G -deflation and i_n is a G -inflation, furthermore, $Z^n(X) \xrightarrow{i_n} X^{n+1} \xrightarrow{p_{n+1}} Z^{n+1}(X)$ is G -exact.

In [8], J. Gillespie constructed the G -derived category $\mathcal{D}(G)$ of the exact category \mathcal{G}_G as the homotopy category of a model structure on the category $\text{Ch}(\mathcal{G})_G$. Let

$$\mathcal{S}_G = \{D_n(G_s), S_n(G_s) \mid s \in S, n \in \mathbb{Z}\}$$

be a set of objects in $\text{Ch}(\mathcal{G})_G$ where $\{G_s\}_{s \in S}$ is the set of generators of \mathcal{G} . The key step of his construction is to prove the cotorsion pair $({}^{\perp_1}(\mathcal{S}_G^{\perp_1}), \mathcal{S}_G^{\perp_1})$ (the “ \perp_1 ” here is taken with respect to $G\text{-Ext}_{\text{Ch}(\mathcal{G})}^1$) is complete which can be obtained by Theorem 3.7 directly.

In fact, let

$$I = \{0 \rightarrow D_n(G_s)\}_{s \in S} \cup \{S_{n+1}(G_s) \rightarrow D_n(G_s)\}_{s \in S}.$$

Then I is a set of degreewise-split G -inflations in $\text{Ch}(\mathcal{G})_G$. Since $\text{Ch}(\mathcal{G})$ is also a Grothendieck category; see for example [23, Lemma 1.1], we know that $\text{Ch}(\mathcal{G})_G$ contains small colimits, and every object in $\text{Ch}(\mathcal{G})_G$ is small; see [13, Proposition A.2]. In particular, I admits the small object argument (see Definition 2.5) with $\text{Cok}I = \mathcal{S}_G$.

Corollary 4.2. [8, Theorem 4.6] *Let \mathcal{G} be a Grothendieck category with a chosen set of generators $\{G_s\}_{s \in S}$. Let I and \mathcal{S}_G be defined as above. Then*

- (a) *The cotorsion pair $({}^{\perp_1}(\mathcal{S}_G^{\perp_1}), \mathcal{S}_G^{\perp_1})$ is a complete in $\text{Ch}(\mathcal{G})_G$.*
- (b) *The subcategory ${}^{\perp_1}(\mathcal{S}_G^{\perp_1})$ of $\text{Ch}(\mathcal{G})_G$ consists precisely of direct summands of \mathcal{S}_G -filtered objects.*

Proof. Since \mathcal{G} is a Grothendieck category, the exact category $\text{Ch}(\mathcal{G})_G$ is Weakly idempotent complete. By construction of I , it is straightforward to verify that I satisfies the conditions of Theorem 3.7, thus the assertions follows from Theorem 3.7(b) directly by noting that $\text{Cell}(I) \subseteq \mathcal{S}_G\text{-Filt}$ (see Lemma 3.3(b)).

□

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